JOURNAL OF FUNCTIONAL ANALYSIS 65, 293–308 (1986)

QC-Level Sets and Quotients of Douglas Algebras

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Communicated by Ralph S. Phillips

Received March 8, 1984; revised May 16, 1985

There is a QC-level set which coincides with a support set (under the continuum hypothesis), and there are Douglas algebras B_1 and B_2 satisfying: both the unit balls of L^∞/B_1 and $B_2/H^\infty + C$ have extreme points but not have exposed points.

We denote by H^∞ the algebra of bounded analytic functions on the unit disk D. Identifying with their boundary functions, we regard H^∞ as the (essentially) uniformly closed subalgebra of L^∞, the space of bounded measurable functions on the unit circle ∂D with respect to the normalized Lebesgue measure m. A uniformly closed subalgebra between H^∞ and L^∞ is called a Douglas algebra. Throughout this paper, we use the capital letter B for a Douglas algebra. $M(B)$ denotes the maximal ideal space of B. We put $X = M(L^\infty)$. Then X is the Shilov boundary for every Douglas algebra. For a point x in $M(H^\infty)$, we denote by μ_x the representing measure on X for x, and by supp μ_x the support set of μ_x. It is well known that $H^\infty + C$ is the smallest Douglas algebra contining H^∞ properly and $M(H^\infty + C) = M(H^\infty)\backslash D$, where C is the space of continuous functions on ∂D([20]). By Chang–Marshall's theorem [5, 18], structures of Douglas algebras are completely determined by inner functions. So it is important to know properties of each inner function. For a function f in L^∞, we put $N(f)$ the closure of the union set of supp μ_x such that $x \in M(H^\infty + C)$ and $f_{|\mathrm{supp}\,\mu_x} \notin H^\infty{}_{|\mathrm{supp}\,\mu_x}$. Roughly speaking, $N(f)$ is a set on which f does not have the analyticity.

Our purpose of this paper is to study properties of $N(\bar{I})$ for inner functions I, and applies them to study QC-level sets and quotients of Douglas algebras. Here $QC = (H^\infty + C) \cap \overline{(H^\infty + C)}$, and $\{x \in X; f(x) = f(x_0) \text{ for } f \in QC\}$ for a point x_0 in X is called a QC-level set (see [21]).

In Section 1, we will give some basic properties of $N(\bar{I})$ for inner functions I. In Theorem 1, we will prove that $N(\bar{I})$ is a weak peak set for

0022-1236/86 $3.00

$QA = H^\infty \cap QC$ and $N(\bar{I})$ does not contain any closed G_δ-subsets of X. Moreover if I is an interpolating or sparse Blaschke product, $N(\bar{I})$ has additional properties (Lemmas 5 and 8). These additional properties are essential points for applications.

In Section 2, we will study extreme and exposed points in unit balls of quotients of Douglas algebras. These objects are extensively studied recently [13, 14, 16, 17, 24]. In Theorem 2, we will prove that if $N(B)$ contains a closed G_δ-subset of X then $\mathrm{ball}(B/H^\infty + C)$ does not have any extreme points. Theorem 2 is a dual version of [13, Theorem 3] and a generalization of [16, Theorem 4]. In Theorems 3 and 4, using the property of $N(\bar{b})$ for a sparse Blaschke product b, we will see that there are Douglas algebras B_1 and B_2 such that both $\mathrm{ball}(L^\infty/B_1)$ and $\mathrm{ball}(B_2/H^\infty + C)$ have extreme points but do not have any exposed points.

In Section 3, we will study QC-level sets. For a point x in $M(H^\infty + C)$, it is known that $\mathrm{supp}\, \mu_x$ is an antisymmetric set for $H^\infty + C$ [3, p. 137], and it is easy to see that an antisymmetric set for $H^\infty + C$ is contained in a QC-level set. In [20, Result 1], Sarason proved that there is a QC-level set which is not antisymmetric for $H^\infty + C$. More precisely, Gorkin [9, Theorem 2.13] proved that there is a QC-level set which contains properly a one point maximal antisymmetric set. For a function f in $H^\infty + C$, we put $Z(f) = \{x \in M(H^\infty + C); f(x) = 0\}$. In Theorem 5, we will prove that if x is a P-point in $Z(b)$ for an interpolating Blaschke product b, $\mathrm{supp}\, \mu_x$ is a QC-level set. If we assume the continum hypothesis, it is known that there is a dense subset of P-points in $Z(b)$ [8, p. 100]. In Theorem 6, we will prove that if x is a cluster point of a countable sequence in $Z(b)$ for a sparse Blaschke product b, then $\mathrm{supp}\, \mu_x$ is not a QC-level set.

We shall give here some notations and definitions. Let $\{z_n\}_{n=1}^\infty$ be a sequence in D. We denote by $\mathrm{cl}(\{z_n\}_{n=1}^\infty)$ the weak*-closure of $\{z_n\}_{n=1}^\infty$ in $M(H^\infty)$. $\{z_n\}_{n=1}^\infty$ is called interpolating and sparse if

$$\inf_n \prod_{m:m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_n z_m} \right| > 0 \qquad \text{and} \qquad \lim_{n \to \infty} \prod_{m:m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_n z_m} \right| = 1,$$

respectively. A Blaschke product is called interpolating and sparse if its zero sequence is interpolating and sparse, respectively. If b is an interpolating Blaschke product with zeros $\{z_n\}_{n=1}^\infty$, then $Z(b) = \mathrm{cl}(\{z_n\}_{n=1}^\infty) \setminus \{z_n\}_{n=1}^\infty$ and $\mathrm{cl}(\{z_n\}_{n=1}^\infty)$ is homeomorphic to the Čech compactification of a countable sequence [11, p. 205]. A point x in $Z(b)$ is called a P-point if for every continuous function f on $Z(b)$ there is an open neighborhood U of x such that $f_{|U}$ is constant [8, p. 63]. For a function f in L^∞, $\|f\|$ means the supremum norm of f, and $\bar{f}$ means the complex conjugate function of f. A function I in H^∞ with $|I| = 1$ is called inner, and we put $N_0(\bar{I})$ the closure of $\bigcup\{\operatorname{supp}\mu_x; x \in Z(I)\}$. Then $N_0(\bar{I})$ is contained in $N(\bar{I})$.

Let A be a closed subalgebra on X and let E be a closed subset of X. E is called a peak set for A if there is a function f in A such that $f=1$ on E and $|f|<1$ on $X\backslash E$, and such a function f is called a peaking function for E. If E is an intersection of some peak sets, E is called a weak peak set. We call E an antisymmetric set for A if there are no nonconstant real functions in $A_{|E}$. For a measure μ on X, we denote by $\mu \perp A$ if $\int_X f\,d\mu = 0$ for every f in A. We denote by $\hat{m}$ the lifting measure of m from ∂D onto X:

$$\int_X f\,d\hat{m} = \int_{\partial D} f\,dm \qquad \text{for every } f \text{ in } L^\infty.$$

Let Y be a Banach space. We denote by ball(Y) the closed unit ball of Y. A point y in ball(Y) is called extreme if $\|y \pm z\| \leqslant 1$ and $z \in Y$ imply $z=0$. A point y in ball(Y) is called exposed if there is a linear functional f of Y such that $\|f\| = f(y) = 1$ and $f(z) \neq 1$ for every z in ball(Y) with $z \neq y$.

For a Douglas algebra B, we denote by $N(B)$ the closure of $\bigcup\{\operatorname{supp}\mu_x; x \in M(H^\infty + C)\backslash M(B)\}$. For a subset F of L^∞, we denote by $[F]$ the closed subalgebra generated by F.

1. N-Sets for Inner Functions

In this section, we study various properties of $N(\bar{I})$ for inner functions I. The following lemma is proved by Sarason [21, Theorem 5].

Lemma 1. *Let f and g be functions in L^∞. If for each point x in $M(H^\infty + C)$ either $f_{|\operatorname{supp}\mu_x} \in H^\infty_{|\operatorname{supp}\mu_x}$ or $g_{|\operatorname{supp}\mu_x} \in H^\infty_{|\operatorname{supp}\mu_x}$, then for each QC-level set Q either $f_{|Q} \in H^\infty_{|Q}$ or $g_{|Q} \in H^\infty_{|Q}$.*

Also Sarason proved the following lemma (unpublished). Gorkin gave a different proof in [9, Theorem 2.8].

Lemma 2. *Let U be a closed and open subset of X and let Q be a QC-level set. We denote by χ_U the characteristic function for U. If $\chi_{U|Q}$ is contained in $H^\infty_{|Q}$, then $\chi_{U|Q}$ is constant.*

Lemma 3. *If E is a closed G_δ-subset of X, then there is a sparse Blaschke product b such that $N(\bar{b})$ is contained in E.*

Proof. Since E is a G_δ-set, there is a function f in L^∞ such that

$$f = 1 \text{ on } E \text{ and } |f| < 1 \text{ on } X\backslash E.$$

We regard f as a continuous function on $M(H^\infty)$ as follows.

$$f(x)=\int_X f\,d\mu_x \qquad \text{for every } x\in M(H^\infty).$$

Then there is a sequence $\{z_n\}_{n=1}^\infty$ in D so that $f(z_n)\to 1$ $(n\to\infty)$. Choosing a subsequence, we may assume that $\{z_n\}_{n=1}^\infty$ is a sparse sequence. Let b be the sparse Blaschke product with zeros $\{z_n\}_{n=1}^\infty$. Since $Z(b)=\mathrm{cl}(\{z_n\}_{n=1}^\infty)\backslash\{z_n\}_{n=1}^\infty$, we have $f=1$ on $Z(b)$. Since $\|f\|\leqslant 1, f=1$ on $\operatorname{supp}\mu_x$ for x in $Z(b)$. This implies that $N_0(\bar{b})$ is contained in E. By the sparseness condition, if y is a point in $M(H^\infty+C)$ with $|b(y)|<1$, then there is a point y_0 in $Z(b)$ such that $y\in P(y_0)$, where $P(y_0)$ is the Gleason part containing y_0 (see the proof of [10, Lemma 1]). By [6, p. 143], $y\in P(y_0)$ implies that μ_y and μ_{y_0} are mutually absolutely continuous. Thus we get $N(\bar{b})=N_0(\bar{b})$, and $N(\bar{b})$ is contained in E.

Remark 1. If b is a sparse Blaschke product, then $N(\bar{b})=N_0(\bar{b})$.

For each x in $M(H^\infty+C)$, there is a unique point $\pi_1(x)$ in $M(QC)$ such that $f(\pi_1(x))=f(x)$ for every f in QC. Then the map π_1 is continuous from $M(H^\infty+C)$ onto $M(QC)$. We put $\pi_0=\pi_{1|X}$, then π_0 is also a continuous map from X onto $M(QC)$. We note that $\{y\in X; \pi_0(y)=\pi_0(x)\}$ is a QC-level set. We denote it by Q_x, that is, $Q_x=\pi_0^{-1}(\pi_1(x))$. Then $\operatorname{supp}\mu_x$ is contained in Q_x. There is a unique probability measure $\hat{m}_0$ on $M(QC)$ such that

$$\int_{M(QC)} f\,d\hat{m}_0=\int_{\partial D} f\,dm \qquad \text{for every } f \text{ in } QC.$$

We note that $\hat{m}_0$ coincides with the measure $\pi_0(\hat{m})$ which is the image of $\hat{m}$ mapped by π_0. Then $\hat{m}(\pi_0^{-1}(E))=\hat{m}_0(E)$ for every closed subset E of $M(QC)$.

THEOREM 1. *If I is a noncontinuous inner function, then*

(i) $N(\bar{I})=\pi_0^{-1}(\pi_1(Z(I)))=\bigcup\{Q_x; x\in Z(I)\}$.

(ii) *$N(\bar{I})$ is the smallest weak peak set for QA containing $N_0(\bar{I})$.*

(iii) *$N(\bar{I})$ does not contain any closed G_δ-subset of X.*

Proof. (i) Let x be a point in $M(H^\infty+C)$ with $|I(x)|<1$. Since $\operatorname{supp}\mu_x$ is a weak peak set for H^∞ [11, p. 207],

$$B=\{f\in L^\infty; f_{|\operatorname{supp}\mu_x}\in H^\infty_{|\operatorname{supp}\mu_x}\}$$

is a proper Douglas algebra. Since $\bar{I}\notin B$, there is a point x_0 in $M(H^\infty+C)$ such that

$$x_0\in Z(I) \qquad \text{and} \qquad \operatorname{supp}\mu_{x_0}\subset\operatorname{supp}\mu_x.$$

This implies that $\operatorname{supp}\mu_x \subset Q_{x_0} = \pi_0^{-1}(\pi_1(x_0))$, and consequently $\operatorname{supp}\mu_x \subset \pi_0^{-1}(\pi_1(Z(I))$ for every $x \in M(H^\infty + C)$ with $|I(x)| < 1$. Since $\pi_0^{-1}(\pi_1(Z(I))$ is a closed subset, we get

$$N(\bar{I}) \subset \pi_0^{-1}(\pi_1(Z(I)).$$

To see (i), suppose that $N(\bar{I}) \subsetneqq \pi_0^{-1}(\pi_1(Z(I)))$. Then there is a QC-level set Q such that

(1) $Q \not\subset N(\bar{I})$ and

(2) $Q \subset \pi_0^{-1}(\pi_1(Z(I)))$.

By (2), there is a point y in $Z(I)$ satisfying $Q = Q_y$. Since $\operatorname{supp}\mu_y \subset Q_y$, we get

(3) $\bar{I}_{|Q} \notin H^\infty_{|Q}$ and

(4) $Q \cap N(\bar{I}) \neq \varnothing$.

By (1), there is a closed and open subset U of X such that

(5) $U \cap Q \neq \varnothing$ and $U \cap N(\bar{I}) = \varnothing$.

By (4) and (5), $\chi_{U|Q}$ is not a constant function. By Lemma 2, we have

(6) $\chi_{U|Q} \notin H^\infty_{|Q}$.

Since $U \cap N(\bar{I}) = \varnothing$,

$$\bar{I}_{|\operatorname{supp}\mu_\zeta} \in H^\infty_{|\operatorname{supp}\mu_\zeta} \qquad \text{or} \qquad \chi_{U|\operatorname{supp}\mu_\zeta} \in H^\infty_{|\operatorname{supp}\mu_\zeta}$$

for every ζ in $M(H^\infty + C)$. By Lemma 1,

$$\bar{I}_{|Q} \in H^\infty_{|Q} \qquad \text{or} \qquad \chi_{U|Q} \in H^\infty_{|Q}.$$

But this contradicts with (3) or (6). Thus we get (i).

(ii) By (i), we have

(7) $\pi_0^{-1}(\pi_0(N(\bar{I}))) = N(\bar{I})$.

We shall see that

(8) $N(\bar{I})$ is a weak peak set for QA.

By Wolff's theorem [22, Theorem 1], there is a nonzero function q in QA such that $q\bar{I} \in QC$. Since a function in QC is constant on $\operatorname{supp}\mu_\zeta$ for every $\zeta \in M(H^\infty + C)\setminus X$ [19, Corollary 3], we have that $q = 0$ on $N(\bar{I})$. Since $q \neq 0$ and $q \in QA$, $\{x \in X; q(x) = 0\}$ is a set of $\hat{m}$-measure zero. Then $\hat{m}_0(\pi_0(N(\bar{I}))) = 0$. By [22, Lemma 2.3], $\pi_0(N(\bar{I}))$ is a weak peak set for QA. Thus we get (8).

To complete the proof of (ii), let E be a weak peak set for QA containing $N_0(\bar{I})$. To prove $N(\bar{I}) \subset E$, suppose that $N(\bar{I}) \not\subset E$. By (7) and $\pi_0^{-1}(\pi_0(E)) = E$, there is a QC-level set Q such that

$$Q \subset N(\bar{I}) \qquad \text{and} \qquad Q \cap E = \varnothing.$$

By (i), there is a point x in $Z(I)$ with $Q_x = Q$. Then $\operatorname{supp} \mu_x \subset Q$, and then $\operatorname{supp} \mu_x \subset Q \cap E$. But this is a contradiction, and we get (ii).

(iii) Suppose that there is a closed G_δ-subset E of X such that $E \subset N(\bar{I})$. Then for some λ with $|\lambda| = 1$, $E_\lambda = \{x \in E; I(x) = \lambda\}$ is a nonempty closed G_δ-subset of X. By Lemma 3, there is a sparse Blaschke product b such that

(9) $N(\bar{b}) \subset E_\lambda \subset N(\bar{I})$.

By the definition of E_λ, there are no points x in $Z(I)$ with $\operatorname{supp} \mu_x \subset E_\lambda$. By (i), there are no QC-level sets Q with $Q \subset E_\lambda$. Since $N(\bar{b})$ contains QC-level sets, this contradicts with (9). This completes the proof of Theorem 1.

COROLLARY 1. *No point in $M(QC)$ is a peak point for QA.*

Proof. Suppose that a point x in $M(QC)$ is a peak point for QA. Then $\pi_0^{-1}(x)$ is a $G_\delta QC$-level set. By Lemma 3, there are two sparse Blaschke products b_1 and b_2 such that $N(\bar{b}_1) \cup N(\bar{b}_2) \subset \pi_0^{-1}(x)$ and $N(\bar{b}_1) \cap N(\bar{b}_2) = \varnothing$. By Theorem 1, $\pi_0^{-1}(x)$ contains at least two QC-level sets, but this is a contradiction.

COROLLARY 2. *If I is an inner function and Q is a QC-level set with $Q \subset N(\bar{I})$, then $\bar{I}_{|Q} \notin H^\infty_{|Q}$.*

Proof. This follows from (i) of Theorem 1.

COROLLARY 3. *Let I_1 and I_2 be inner functions. If for each point x in $M(H^\infty + C)$ either $\bar{I}_{1|\operatorname{supp}\mu_x} \in H^\infty_{|\operatorname{supp}\mu_x}$ or $\bar{I}_{2|\operatorname{supp}\mu_x} \in H^\infty_{|\operatorname{supp}\mu_x}$, then $N(\bar{I}_1) \cap N(\bar{I}_2) = \varnothing$.*

Proof. Suppose that $N(\bar{I}_1) \cap N(\bar{I}_2) \neq \varnothing$. By Theorem 1(i), there is a QC-level set Q such that $Q \subset N(\bar{I}_1) \cap N(\bar{I}_2)$. By Corollary 2, $\bar{I}_{1|Q} \notin H^\infty_{|Q}$ and $\bar{I}_2 \notin H^\infty_{|Q}$. But this contradicts with the assertion of Lemma 1.

COROLLARY 4. *Let B be a Douglas algebra. If I is an inner function with $N(\bar{I}) \supset N(B)$, then $B \subset [H^\infty, \bar{I}]$.*

Proof. Suppose that $B \not\subset [H^\infty, \bar{I}]$. By the Chang–Marshall theorem [5, 18], there is an inner function I_1 such that $\bar{I}_1 \in B$ and $[H^\infty, \bar{I}_1] \not\subset [H^\infty, \bar{I}]$. Then $M([H^\infty, \bar{I}_1]) \not\supset M([H^\infty, \bar{I}])$ and $N(\bar{I}_1) \subset N(B) \subset N(\bar{I})$. Since $M([H^\infty, \bar{I}]) = \{x \in M(H^\infty + C);\ |I(x)| = 1\}$, there is a point x_0 in $M(H^\infty + C)$ with $|I(x_0)| = 1$ and $|I_1(x_0)| < 1$. By the corona theorem [4], there is a net $\{z_\alpha\}_\alpha$ in D such that $z_\alpha \to x_0$. Let us take a subsequence $\{z_n\}_{n=1}^\infty$ in $\{z_\alpha\}_\alpha$ such that

(1) $I_1(z_n) \to I_1(x_0)$ and $I(z_n) \to I(z_0)$ $(n \to \infty)$.

We may assume that $\{z_n\}_{n=1}^\infty$ is a sparse sequence. Let b be the sparse Blaschke product with zeros $\{z_n\}_{n=1}^\infty$. By (1), we have

(2) $I_1 = I_1(x_0)$ on $Z(b)$ and

(3) $I = I(x_0)$ on $Z(b)$.

Since $|I(x_0)| = 1$, $I = I(x_0)$ on $N_0(\bar{b})$ by (3). Since $N(\bar{b}) = N_0(\bar{b})$ by Remark 1, a pair of inner functions I and b satisfies the assumptions of Corollary 3. Thus we get

(4) $N(\bar{b}) \cap N(\bar{I}) = \emptyset$.

Since $|I_1(x_0)| < 1$, by (2) we have

(5) $N(\bar{I}_1) \supset N_0(\bar{b}) = N(\bar{b})$;

(4) and (5) give us $N(\bar{I}_1) \not\subset N(\bar{I})$, but this contradicts with $N(\bar{I}_1) \subset N(\bar{I})$. Thus we get $B \subset [H^\infty, \bar{I}]$.

By Corollary 4, we get the following two corollaries.

Corollary 5. *Let I_1 and I_2 be inner functions, then $[H^\infty, \bar{I}_1] = [H^\infty, \bar{I}_2]$ if and only if $N(\bar{I}_1) = N(\bar{I}_2)$.*

Corollary 6. *Let B be a Douglas algebra and let I be an inner function with $\bar{I} \in B$. Then $B = [H^\infty, \bar{I}]$ if and only if $N(B) = N(\bar{I})$.*

The following corollary will be used to prove Theorem 2.

Corollary 7. *For a sequence $\{f_n\}_{n=1}^\infty$ in L^∞, $\mathrm{cl}(\bigcup\{N(f_n); n = 1, 2, \ldots\})$ does not contain any G_δ-subset of X.*

Proof. By [15, Lemma 2.2], there is a Blaschke product b such that $h_n = bf_n \in H^\infty + C$ for every n. Since $f_n = \bar{b}h_n$, if $f_{n|\mathrm{supp}\,\mu_x} \notin H^\infty_{|\mathrm{supp}\,\mu_x}$ for $x \in M(H^\infty + C)$ then $\bar{b}_{|\mathrm{supp}\,\mu_x} \notin H^\infty_{|\mathrm{supp}\,\mu_x}$. Thus $N(f_n) \subset N(\bar{b})$ for every n. By Theorem 1(iii), we get our assertion.

2. Quotients of Douglas Algebras

In this section, we study a problem that which Douglas algebras B, $\mathrm{ball}(B/H^\infty + C)$ $(\mathrm{ball}(L^\infty/B))$ has extreme or exposed points.

LEMMA 4. *Let f be a function in B with $1 = \|f\| = \|f + H^\infty + C\|$. If there is a QC-level set Q such that $N(f) \cap Q = \emptyset$ and $N(B) \cap Q \neq \emptyset$, then $f + H^\infty + C$ is not an extreme point of* $\mathrm{ball}(B/H^\infty + C)$.

Proof. By our assumption, there is a function q in QC such that

$$0 \leqslant q \leqslant 1, \quad q = 1 \text{ on } Q \qquad \text{and} \qquad q = 0 \text{ on } N(f).$$

Then $fq \in H^\infty + C$. Let U be an open subset of X such that $U = \pi_0^{-1}(\pi_0(U))$, $U \supset Q$, and $q \geqslant \varepsilon$ on U for some $\varepsilon > 0$. Since $N(B) \cap Q \neq \emptyset$ and $N(B)$ is the closure of

$$\bigcup \{N(\bar{I});\ I \text{ is inner with } \bar{I} \in B\},$$

there is an inner function I such that

$$\bar{I} \in B \qquad \text{and} \qquad N(\bar{I}) \cap U \neq \emptyset.$$

Let us take a QC-level set Q_1 with $Q_1 \subset N(\bar{I} \cap U)$. By Corollary 2, $\bar{I}_{|Q_1} \notin H^\infty_{|Q_1}$. Since $q_{|Q_1}$ is nonzero constant, we get $\bar{I}q_{|Q_1} \notin H^\infty_{|Q_1}$, so that $\bar{I}q \notin H^\infty + C$. Since $\bar{I}q \in B$,

$$\bar{I}q + H^\infty + C \in B/H^\infty + C \qquad \text{and} \qquad \bar{I}q + H^\infty + C \neq H^\infty + C.$$

Then the following inequalities imply that $f + H^\infty + C$ is not an extreme point of $\mathrm{ball}(B/H^\infty + C)$.

$$\begin{aligned} &\|f + H^\infty + C \pm (\bar{I}q + H^\infty + C)\| \\ &\leqslant \|f(1-q) \pm \bar{I}q\| \\ &\leqslant \sup_{x \in X} \{|1 - q(x)| + |q(x)|\} = 1. \end{aligned}$$

In [13, Theorem 3], it is proved that if the essential set for B contains a closed G_δ-subset of X then $\mathrm{ball}(L^\infty/B)$ does not have any extreme point. Here a closed subset Γ of X is called the essential set for B if Γ is the smallest closed subset such that if $f_{|\Gamma} = 0$ and $f \in L^\infty$ imply $f \in B$. The following theorem is the dual version of the above result. Also in [16], we proved that if $\hat{m}(N(B)) > 0$ then $\mathrm{ball}(B/H^\infty + C)$ does not contain any extreme points. If $\hat{m}(N(B)) > 0$, then $N(B)$ contains a closed and open subset of X [6, p. 18]. So Theorem 2 is a generalization of the above result.

THEOREM 2. *Let B be a Douglas algebra with $B \supsetneqq H^\infty + C$. If $N(B)$ contains a closed G_δ-subset of X, then* $\mathrm{ball}(B/H^\infty + C)$ *does not have any extreme points.*

Proof. Suppose that there is a closed G_δ-subset E of X such that $E \subset N(B)$. Let $f \in B$ with $1 = \|f + H^\infty + C\|$. Since $H^\infty + C$ has the best approximation property [1], we may assume that $\|f\| = 1$. We note that $N(f) \subset N(B)$. By Corollary 7, we have $E \not\subset N(f)$. Let us take a closed and open subset U of X such that $E \cap U \neq \emptyset$ and $N(f) \cap U = \emptyset$. By Lemma 3 and Theorem 1, (also [9, Theorem 2.13]), there is a QC-level set Q in $E \cap U$. By Lemma 4, $f + H^\infty + C$ is not an extreme point of ball$(B/H^\infty + C)$.

The following proposition shows that there are many Douglas algebras satisfying the assumptions of Theorem 2.

PROPOSITION 1. *For each closed G_δ-subset E of X, there is a Douglas algebra B such that $N(B) = E$.*

Proof. We put

$$B = [H^\infty, \bar{b}; \{b \text{ is a sparse Blaschke product with } N(\bar{b}) \subset E\}].$$

Then $N(B) \subset E$. To see $N(B) = E$, suppose that $N(B) \subsetneqq E$. Since $E \backslash N(B)$ contains a closed G_δ-subset of X, there is a sparse Blaschke product b_0 such that $N(\bar{b}_0) \subset E \backslash N(B)$ by Lemma 3. By the definition of B, $\bar{b}_0 \in B$ and $N(\bar{b}_0) \subset N(B)$. But this is a contradiction.

According to [13, Theorem 3] and Theorem 2, to answer the problem which is mentioned in the beginning of this section we have to investigate the cases that the essential set for B or $N(B)$ does not contain any G_δ-subsets. We guess that it is difficult to give a complete answer. In Theorems 3 and 4, we will give partial answers.

THEOREM 3. *Let b be an interpolating Blaschke product. We put $B = \{f \in L^\infty; f_{|N(\bar{b})} \in H^\infty_{|N(\bar{b})}\}$. Then B is a Douglas algebra and $\bar{b} + B$ is an extreme point of ball(L^∞/B). Moreover if b is a sparse Blaschke product, then* ball(L^∞/B) *does not have any exposed points.*

To prove Theorem 3, we need some lemmas. The following lemma is a consequence of the sparseness condition.

LEMMA 5. *Let b be a sparse Blaschke product with zeros $\{z_n\}_{n=1}^\infty$. If x and y are distinct points in $Z(b)$, then $Q_x \neq Q_y$.*

Proof. Let x and y are distinct points in $Z(b)$. There are sparse Blaschke products b_1 and b_2 such that

$$b = b_1 b_2, \qquad b_1(x) = 0, \qquad \text{and} \qquad b_2(y) = 0.$$

Then we have $Z(b_1) \cap Z(b_2) = \varnothing$. Suppose that there is a point ζ in $M(H^\infty + C)$ such that $|b_1(\zeta)| < 1$ and $|b_2(\zeta)| < 1$. By the proof of Lemma 3, there are two points ζ_1 and ζ_2 in $P(\zeta)$ such that $b_1(\zeta_1) = 0$ and $b_2(\zeta_2) = 0$. Then b has two zero points in $P(\zeta)$ counting zero's multiplicity. By [12, p. 107], b has only one zero point in $P(\zeta)$. But this is a contradiction. Then b_1 and b_2 satisfy the assumptions of Corollary 3. Thus we get

$$Q_x \cap Q_y \subset N(\bar{b}_1) \cap N(\bar{b}_2) = \varnothing.$$

The following lemma is a characterization of exposed points of $\mathrm{ball}(L^\infty/B)$.

LEMMA 6 [14, Theorem 1]. *If a Douglas algebra B has the best approximation property, then* $\mathrm{ball}(L^\infty/B)$ *has exposed points if and only if there is a measure μ on X such that $\mu \perp B$ and* $\operatorname{supp} \mu$ *coincides with the essential set for B.*

The following is proved in [2, 10] independently. For f in B, we put

$$Z_B(f) = \{x \in M(B); f(x) = 0\}.$$

LEMMA 7. *Let B be a Douglas algebra and let I be an interpolating Blaschke product. If a function f in B satisfies $Z_B(f) \supset Z_B(I)$, then $f\bar{I} \in B$.*

LEMMA 8 (cf. Theorem 1(ii)). *If b is an interpolating Blaschke product, then $N(\bar{b})$ is the smallest weak peak set for $H^\infty + C$ contining $N_0(\bar{b})$.*

Proof. Let E be a peak subset of X for $H^\infty + C$ containing $N_0(\bar{b})$. Then there is a function f in $H^\infty + C$ such that

$$\|f\| = 1, \qquad f = 1 \text{ on } E \qquad \text{and} \qquad |f| < 1 \text{ on } X \setminus E.$$

We note that $1 - f \in H^\infty + C$ and $1 - f = 0$ on E. Since $E \supset N_0(\bar{b})$, we have $Z(1 - f) \supset Z(b)$. By Lemma 7,

$$(1 - f)\,\bar{b} \in H^\infty + C \qquad \text{and} \qquad (1 - f)\,\bar{b} = 0 \text{ on } E.$$

Repeating this argument, we get

$$(1 - f)\,\bar{b}^n \in H^\infty + C \qquad \text{for every } n = 1, 2, \ldots.$$

Since $\|(1 - f)\,\bar{b}^n\| \leqslant 2$, we have

$$1 - f = 0 \qquad \text{on } \{x \in M(H^\infty + C);\ |b(x)| < 1\}.$$

Then $\int_X f\, d\mu_x = 1$ for every $x \in M(H^\infty + C)$ with $|b(x)| < 1$. Since $\|f\| \leqslant 1$, $f = 1$ on $\operatorname{supp} \mu_x$. As a consequence, $f = 1$ on $N(\bar{b})$. This implies that $E \supset N(\bar{b}) \supset N_0(\bar{b})$. By Theorem 1(ii), we get our assertion.

Proof of Theorem 3. By Theorem 1, $N(\bar{b})$ is a weak peak set for QA, so that B is a Douglas algebra. By the definition of B, $\{x \in M(H^\infty + C); |b(x)| < 1\} \subset M(B)$. Then we have $\|\bar{b} + B\| = \|1 + bB\| = 1$. We shall prove that

(1) $\bar{b} + B$ is an extreme point of ball(L^∞/B).

Let f be a function in L^∞ such that

(2) $\|\bar{b} + B \pm (f + B)\| \leqslant 1$.

To see (1), it is sufficient to prove that

(3) $f + B = B$, that is, $f \in B$.

By [23, Corollary 3.2], B has the best approximation property. So there are two functions g and h in B such that

(4) $1 \geqslant \|\bar{b} + f + g\|$ and $1 \geqslant \|\bar{b} - f + h\|$.

We put $F = (g + h)/2$, then $F \in B$ and

$$\|\bar{b} + F\| \leqslant (\|\bar{b} + f + g\| + \|\bar{b} - f + h\|)/2 \leqslant 1.$$

Consequently, we have $\|1 + bF\| \leqslant 1$. Since $Z(b) \subset M(B)$, $bF = 0$ on $Z(b)$, and then $1 = \int_X (1 + bF)\, d\mu_x$ for every $x \in Z(b)$. Since $\|1 + bF\| \leqslant 1$, $bF = 0$ on supp μ_x for $x \in Z(b)$. Thus we get $1 + bF = 1$ on $N_0(\bar{b})$. We shall see

(5) $1 + bF = 1$ on $N(\bar{b})$.

Since $N(\bar{b})$ is a weak peak set for H^∞ and $(1 + bF)_{|N(\bar{b})} \in H^\infty_{|N(\bar{b})}$, there is a function G in H^∞ such that $G = 1 + bF$ on $N(\bar{b})$ and $\|G\| \leqslant \|(1 + bF)_{|N(\bar{b})}\| = 1$ [6, p. 58]. We put $G_0 = (1 + G)/2$, then $\|G_0\| = 1$ and $G_0 = 1$ on $N_0(\bar{b})$. Since $\{x \in M(H^\infty + C);\ G_0(x) = 1\}$ is a peak set for H^∞, we get $\{x \in M(H^\infty + C);\ G_0(x) = 1\} \supset N(\bar{b})$ by Lemma 8. Thus we get (5).

By (5), we have $h = -g$ on $N(\bar{b})$. Then (4) implies that

$$1 \geqslant |\bar{b} \pm (f + g)| \qquad \text{on } N(\bar{b}).$$

Consequently we have $f + g = 0$ on $N(\bar{b})$. Since $g_{|N(\bar{b})} \in H^\infty_{|N(\bar{b})}$, we get (3). This completes the proof of the first assertion.

Next we shall prove the second assertion. We assume that b is a sparse Blaschke product with zeros $\{z_n\}_{n=1}^\infty$. We will prove that there are no positive measures μ on $N(\bar{b})$ with supp $\mu = N(\bar{b})$. Then the proof will be completed by Lemma 6. Let μ be a positive measure on $N(\bar{b})$. By Lemma 5,

(6) $\pi_1(Z(b))$ and $Z(b)$ are homeomorphic.

By Theorem 1(i), π_0 maps $N(\bar{b})$ onto $\pi_1(Z(b))$, so that $\pi_0(\mu)$ is the positive measure on $\pi_1(Z(b))$ which is the image measure of μ by the map π_0. Since $Z(b) = \mathrm{cl}(\{z_n\}_{n=1}^{\infty}) \backslash \{z_n\}_{n=1}^{\infty}$ and $\mathrm{cl}(\{z_n\}_{n=1}^{\infty})$ is homeomorphic to the Čech compactification of the discrete sequence $\{z_n\}_{n=1}^{\infty}$ [11, p. 205], every G_δ-subset $Z(b)$ contains a closed and open subset of $Z(b)$. This means that there are no measures on $Z(b)$ with the full support. By (6), we get that $\operatorname{supp} \pi_0(\mu) \subsetneqq \pi_1(Z(b))$, and then $\operatorname{supp} \mu \subsetneqq N(\bar{b})$. This completes the proof.

THEOREM 4. *Let b be a sparse Blaschke product. We put $B = [H^\infty, \bar{b}]$, then*

(i) $\mathrm{ball}(B/H^\infty + C)$ *has extreme points and*

(ii) $\mathrm{ball}(B/H^\infty + C)$ *does not have any exposed points.*

To prove Theorem 4, we need a lemma.

LEMMA 9. *Let B be a Douglas algebra such that $N(B)$ is a weak peak set for $H^\infty + C$. Let $f \in B$ with $1 = \|f\| = \|f + H^\infty + C\|$. If μ is a measure on X with $\mu \perp H^\infty + C$ and $1 = \|\mu\| = \int_X f\, d\mu$, then* $\operatorname{supp} \mu$ *is contained in* $N(B)$.

Proof. Suppose that $\operatorname{supp} \mu \not\subset N(B)$. Since $N(B)$ is a weak peak set for $H^\infty + C$, there is a peak subset E of X for $H^\infty + C$ such that $N(B) \subset E$ and $\operatorname{supp} \mu \not\subset E$. Then $|\mu|(E) < 1$. Let $g \in H^\infty + C$ be a peaking function for E. Since $N(f) \subset N(B) \subset E$, $(1 - g^n) f \in H^\infty + C$ for every n. Since

$$0 = \int_X (1 - g^n) f\, d\mu \to \int_{X \backslash E} f\, d\mu \qquad (n \to \infty),$$

we get $\int_{X \backslash E} f\, d\mu = 0$. Consequently,

$$\begin{aligned} 1 = \int_X f\, d\mu &= \int_E f\, d\mu + \int_{X \backslash E} f\, d\mu \\ &\leqslant \left| \int_E f\, d\mu \right| \leqslant |\mu|(E) < 1. \end{aligned}$$

But this is a contradiction, and we get our assertion.

Proof of Theorem 4. The assertion (i) is proved in [16, Theorem 5] for an interpolating Blaschke product b.

(ii) Let f be a function in B with $\|f + H^\infty + C\| = 1$.
We shall prove that

(1) $f + H^\infty + C$ is not an exposed point of $\mathrm{ball}(B/H^\infty + C)$.

Since $H^\infty + C$ has the best approximation property [1], we may assume that

(2) $\|f\| = \|f + H^\infty + C\| = 1.$

To prove (1), let μ be a measure on X satisfying

(3) $\|\mu\| = 1$, $\mu \perp H^\infty + C$, and $\int_X f \, d\mu = 1$.

By (3) and Lemma 9, $\operatorname{supp} \mu \subset N(B) = N(\bar{b})$. By the proof of Theorem 3, $\operatorname{supp} \pi_0(|\mu|) \subsetneqq \pi_1(Z(b))$. Then there is a point x in $Z(b)$ such that $\pi_1(x) \notin \operatorname{supp} \pi_0(|\mu|)$.

Case 1. $f_{|\operatorname{supp}\mu_x} \notin H^\infty_{|\operatorname{supp}\mu_x}$. There is a function q in QC such that $\|q\| \leqslant 1$, $q = 1$ on $\operatorname{supp} \mu$ and $q(x) = 0$. Then

(4) $fq \in B$, $\|fq + H^\infty + C\| \leqslant 1$ and $\int_X fq \, d\mu = \int_X f \, d\mu = 1$.

Since $(f - fq)_{|\operatorname{supp}\mu_x} = f_{|\operatorname{supp}\mu_x} \notin H^\infty_{|\operatorname{supp}\mu_x}$, we have

(5) $f + H^\infty + C \neq fq + H^\infty + C$.

Then (4) and (5) imply (1).

Case 2. $f_{|\operatorname{supp}\mu_x} \in H^\infty_{|\operatorname{supp}\mu_x}$. Let us take two nonnegative functions q_1 and q_2 in QC such that $q_1 + q_2 \leqslant 1$ on X, $q_1 = 1$ on $\operatorname{supp} \mu$ and $q_2(x) = 1$. We put $g = fq_1 + \bar{b} q_2$, then

(6) $g \in B$ and $\|g + H^\infty + C\| \leqslant \|g\| \leqslant q_1 + q_2 \leqslant 1$.

Since $q_2 = 0$ on $\operatorname{supp} \mu$, we get

(7) $\int_X g \, d\mu = \int_X f \, d\mu = 1$.

Since $q_1 = 0$ on $\operatorname{supp} \mu_x$ and $q_2 = 1$ on $\operatorname{supp} \mu_x$, we have

$$(f - g)_{|\operatorname{supp}\mu_x} = f_{|\operatorname{supp}\mu_x} - \bar{b}_{|\operatorname{supp}\mu_x} \notin H^\infty_{|\operatorname{supp}\mu_x}.$$

Then

(8) $f + H^\infty + C \neq g + H^\infty + C$.

By (6), (7), and (8), we get (1).

In the last part of this section, we shall give a property of annihilating measures for $H^\infty + C$ which follows from Lemma 9 and Theorem 1(iii). In [15, Corollary 5.10], we proved that if $\{\mu_n\}_{n=1}^\infty$ is a sequence of measures on X with $\mu_n \perp H^\infty + C$ then $\hat{m}(\operatorname{cl}(\bigcup\{\operatorname{supp} \mu_n;\ n = 1, 2, \ldots\})) = 0$. The following corollary is a generalization of the above fact.

COROLLARY 8. *If* $\{\mu_n\}_{n=1}^\infty$ *is a sequence of measures on* X *with* $\mu_n \perp H^\infty + C$, *then* $\operatorname{cl}(\bigcup\{\operatorname{supp} \mu_n; n = 1, 2, \ldots\})$ *does not contain any closed* G_δ*-subset of* X.

Proof. We may assume that $\|\mu_n\| = 1$ for every n. By [15, Theorem 2.1], there is a Blaschke product b such that

$$b|\mu_n| \perp H^\infty + C \qquad \text{for every } n.$$

We put $B=[H^\infty, \bar{b}]$. Then $N(B)=N(\bar{b})$ is a weak peak set for QA by Theorem 1(ii). We put $f=\bar{b}$ and $\mu=b\,|\mu_n|$, then f and μ satisfy the assumptions of Lemma 9. Thus we get $\operatorname{supp}\mu_n \subset N(B)=N(\bar{b})$ for every n. By Theorem 1(iii), we get our assertion.

3. Support Sets and QC-Level Sets

In this section, we study $\operatorname{supp}\mu_x$ and Q_x for $x \in Z(b)$, where b is a sparse or interpolating Blaschke product.

THEOREM 5. *If b is an interpolating Blaschke product and x is a P-point in $Z(b)$, then* $\operatorname{supp}\mu_x = Q_x$ *(under the continuum hypothesis, there is a P-point in $Z(b)$).*

Proof. Let $\{z_n\}_{n=1}^\infty$ be the zero sequence in D of b. Let $x \in Z(b)$ be a P-point. To prove $\operatorname{supp}\mu_x = Q_x$, suppose $\operatorname{supp}\mu_x \subsetneqq Q_x$. Since $\operatorname{supp}\mu_x$ is a weak peak set for H^∞, there is a peak set E of X for H^∞ such that

(1) $\operatorname{supp}\mu_x \subset E$ and $Q_x \not\subset E$.

Let us take a function f in H^∞ such that

$$\|f\|=1, \quad f=1 \text{ on } E \qquad \text{and} \qquad |f|<1 \text{ on } X\backslash E.$$

Then $f(x)=\int_X f\,d\mu_x = 1$. Since x is a P-point of $Z(b)$, there is a closed and open subset U of $Z(b)$ such that

$$x \in U \qquad \text{and} \qquad f=1 \text{ on } U.$$

Then there is a subsequence $\{z'_n\}_{n=1}^\infty$ of $\{z_n\}_{n=1}^\infty$ such that $\operatorname{cl}(\{z'_n\}_{n=1}^\infty)\backslash\{z'_n\}_{n=1}^\infty = U$. We put b_1 the interpolating Blaschke product with zeros $\{z'_n\}_{n=1}^\infty$ and $b_2 = b\bar{b}_1$. Then b_2 is an interpolating Blaschke product satisfying

$$b=b_1 b_2, \qquad Z(b_1)=U, \qquad \text{and} \qquad Z(b_2)=Z(b)\backslash U.$$

Let y be a point in $Z(b)$. Then $1=f(y)=\int_X f\,d\mu_y$. Since $\|f\|\leqslant 1$, we get $\operatorname{supp}\mu_y \subset E$. Thus $N_0(\bar{b}_1)\subset E$. By Lemma 8, $N(\bar{b}_1)\subset E$. Since $x \in U = Z(b_1)$, we get $Q_x \subset N(\bar{b}_1) \subset E$. This contradicts with (1), and we get $\operatorname{supp}\mu_x = Q_x$.

Remark 2. We do not know whether there is a point x in $M(H^\infty + C)$ with $\operatorname{supp}\mu_x = Q_x$ or not without assuming the continuum hypothesis.

THEOREM 6. *Let b be a sparse Blaschke product. If a point x in $Z(b)$ is a cluster point of a countable subset of $Z(b)$, then* $\operatorname{supp}\mu_x \subsetneqq Q_x$.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $Z(b)$ such that x is a cluster point of $\{x_n\}_{n=1}^{\infty}$. Since $\int_X b\, d\mu_{x_n} = 0$ and $|b| = 1$, we have $b(\operatorname{supp} \mu_{x_n}) = D$. We take a sequence $\{y_n\}_{n=1}^{\infty}$ such that

(1) $y_n \in \operatorname{supp} \mu_{x_n}$ and $b(y_n) = 1$ for $n = 1, 2, \ldots$.

We shall see

(2) $Q_x \cap \operatorname{cl}(\{y_n\}_{n=1}^{\infty}) \neq \emptyset$.

To see (2), suppose that $Q_x \cap \operatorname{cl}(\{y_n\}_{n=1}^{\infty}) = \emptyset$. Then there is a function f in QC such that

$$f = 1 \text{ on } Q_x \qquad \text{and} \qquad f = 0 \text{ on } \operatorname{cl}(\{y_n\}_{n=1}^{\infty}).$$

This means that $f(x) = 1$ and $0 = f(y_n) = f(x_n)$ for $n = 1, 2, \ldots$. But this contradicts with $x \in \operatorname{cl}(\{x_n\}_{n=1}^{\infty})$, and we get (2).

Next we shall prove

(3) $\operatorname{supp} \mu_x \cap \operatorname{cl}(\{y_n\}_{n=1}^{\infty}) = \emptyset$.

To see (3), we put

$$\mu = \sum_{n=1}^{\infty} (\tfrac{1}{2})^n \delta_{y_n},$$

where δ_{y_n} is the unit point mass at y_n. Since $x_n \neq x$, $Q_{x_n} \cap Q_x = \emptyset$ by Lemma 5. Since $\operatorname{supp} \mu_x \subset Q_x$ and $y_n \in Q_{x_n}$, we get

(4) $\mu(\operatorname{supp} \mu_x) = 0$.

By (1), we have $b = 1$ on $\operatorname{cl}(\{y_n\}_{n=1}^{\infty})$. Since x is a point in $Z(b)$,

$$\int_X ((1+b)/2)^n\, d\mu_x = (\tfrac{1}{2})^n \to 0 \qquad (n \to \infty).$$

Consequently we get

(5) $0 = \mu_x(\operatorname{cl}(\{y_n\}_{n=1}^{\infty})) = \mu_x(\operatorname{supp} \mu)$.

By the Hoffman's result (see [24, Theorem D]), we get

$$\operatorname{supp} \mu_x \cap \operatorname{supp} \mu = \emptyset.$$

This implies (3). By (2) and (3), we get our assertion.

Acknowledgments

Part of this research was done while the author was a Visiting Scholar at the University of California, Berkeley. The author would like to express his thanks to D. Sarason and W. Badé for several helpful conversations.

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